as opposed to complex

4.1 – Real Vector Spaces

Definition: (Vector space)

Let V be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called **scalars**. By **addition** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the **sum** of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k and m, then we call V a **vector space** and we call the objects in V **vectors**.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V. – closure under addition
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There exists an object in V, called the **zero vector**, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a **negative** of $\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is an object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

closure under scalar multiplication

Notes: ① Vector addition and scalar multiplication can be any rules that satisfy these axioms.

② A vector is any object in a vector space.

#6 Determine whether the set equipped with the given operations is a vector space. For those that are not vector spaces, Identify the vector space axioms that fail.

The set of all n-tuples of real numbers that have the form $(x,\ x,\ \dots,\ x)$ with the standard operations on R^n.

Let $\vec{u}=(u,u,\dots,u)$, $\vec{v}=(v,v,\dots,v) \in R^n$, $k \in R$.

① $\vec{u}+\vec{v}=(u+v, u+v, \dots, u+v)$ is in the set ✓

⑥ $k\vec{u}=(ku, ku, \dots, ku)$ is in the set ✓

②, ③, ⑦-⑩ are satisfied by properties given in Thm 3.1.1.

④ $\vec{0}=(0,0,\dots,0)$

⑤ $-\vec{u}=(-u,-u,\dots,-u)$

Examples of Vector Spaces

1. The simplest vector space is $\{\mathbf{0}\}$.
2. R^n with the usual operations of addition and scalar multiplication of n-tuples
3. R^∞, the set of infinite sequences of numbers with the usual operations of addition and scalar multiplication performed componentwise
4. The set of $m \times n$ matrices with matrix addition and scalar multiplication, denoted M_{mn}
5. The set of real-valued functions $f(x)$ that are defined for all $x \in R$, denoted $F(-\infty, \infty)$ with addition and scalar multiplication $(\mathbf{f}+\mathbf{g})(x) = f(x)+g(x)$ and $(k\mathbf{f})(x) = kf(x)$
6. The set of polynomials of degree $\leq n$, denoted P_n

Example: Show why #5 is a vector space.

Example 4.1.7: Not a vector space

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Define $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $k\mathbf{u} = (ku_1, 0)$.

8 : $(k+m)\vec{u} = ((k+m)u_1, 0) = (ku_1 + mu_1, 0)$

holds ✓ $= (ku_1, 0) + (mu_1, 0) = k\vec{u} + m\vec{u}$

10 : $1\vec{u} = (1u_1, 0) = (u_1, 0) \neq \vec{u}$ Fails.

Example 4.1.8: Unusual vector space

Let V be the set of positive real numbers, so $\mathbf{u} = u$ and $\mathbf{v} = v$ are positive real numbers. Define $\mathbf{u} + \mathbf{v} = uv$ and $k\mathbf{u} = u^k$. — scalar mult. is exponentiation

vector addition is mult. of real #s

Then, for instance $2+5 = 10$, $2(5) = 25$

The set is closed under addition and scalar multiplication (both give pos. real #s) ①, ⑥

vector addition is comm. and asscociative because mult. of real #s is. ②, ③

The zero element: $\vec{0} = 1$ because

$\vec{0} + \vec{u} = 1u = u = \vec{u}$ ④ ✓ $\vec{u}+\vec{v} = uv$

⑩ $1\vec{u} = u^1 = u = \vec{u}$ ✓

⑤ $\vec{u} + (-\vec{u}) = \vec{0} \Rightarrow u(\text{thing}) = 1 \Rightarrow -\vec{u} = \frac{1}{u}$

⑦ $k(\vec{u}+\vec{v}) = (uv)^k = u^k v^k = k\vec{u} + k\vec{v}$ ✓

⑧ $(k+m)\vec{u} = u^{k+m} = u^k u^m = k\vec{u} + m\vec{u}$ ✓

⑨ $k(m\vec{u}) = (m\vec{u})^k = (u^m)^k = u^{(mk)} = u^{(km)} = (km)\vec{u}$ ✓

Example: Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 2,\ u_2 + v_2)$, $k\mathbf{u} = (ku_1, ku_2)$.

a. Show that Axioms 4 and 5 hold.

b. Find all axioms that fail to hold.

a) Find $\vec{0} \ni \vec{u} + \vec{0} = \vec{u}$. Let $\vec{0} = (a, b)$

$\vec{u} + \vec{0} = (u_1, u_2) + (a, b) = \boxed{(u_1 + a + 2, u_2 + b) = (u_1, u_2)} = \vec{u}$

Find a, b: $u_1 + a + 2 = u_1 \Rightarrow a = -2$, $u_2 + b = u_2 \Rightarrow b = 0$

$\vec{0} = (-2, 0)$

5 Find $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$

Let $-\vec{u} = (a, b)$.

$\vec{u} + (-\vec{u}) = (u_1 + a + 2, u_2 + b) = (-2, 0) = \vec{0}$

$u_1 + a + 2 = -2 \Rightarrow a = -u_1 - 4$, $u_2 + b = 0 \Rightarrow b = -u_2$

$-\vec{u} = (-u_1 - 4, -u_2)$

7 & 8 will fail. Confirm

Theorem 4.1.1 Let V be a vector space, $\mathbf{u}$ be a vector in V, and k a scalar; then:

a) $0\mathbf{u} = \mathbf{0}$

b) $k\mathbf{0} = \mathbf{0}$

c) $(-1)\mathbf{u} = -\mathbf{u}$

d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$

Example: Prove that the zero vector in any vector space is unique.